OFFICE OF THE REGISTRAR OF ... 18 FEB 1915

THE JOURNAL

OF THE

Indian Mathematical Society.

Vol. VII.] FEBRUARY 1915. [No. 1.

PROGRESS REPORT.

The following gentlemen have been elected members at the concessional rate.—

1. *Mr. A. G. Chandrachud*, B. A.,—Student, Fergusson College, Poona City.
2. *Mr. T. P. Krishnaswami*, B. A.,—Mathematics Assistant, C. S. School, Srivaikuntham.

With the commencement of the year Messrs. E. W. Middlemast and P. V. Seshu Aiyar have duly taken their seats on the Committee, and Mr. R. Swaminatha Iyer retires from the Committee. Mr. Middlemast has also taken over charge of the office of the President the Society from Prof. R. N. Apte. The best thanks of the Committee are due to the retiring gentlemen for the zealous and conscientious discharge of their duties.

The Calcutta University Calendar for 1914, *Part I*, has been received at the Library.

POONA, *31st January, 1915.*

D. D. KAPADIA,
Hony. Joint Secretary.

So-called Cases of Failure in the Solution of Linear Differential Equations.

By Eric H. Neville.

There are two ways in which the solution of a particular linear differential equation may "fail" although the solution of a more general equation obtained by replacing certain constants by parameters is complete. The two ways are distinct, but they may both be illustrated by the equation

$$(D-1)^2 y = e^x, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where D as usual stands for d/dx.

For the general equation

$$(D-l)\,(D-m)\,y = e^{nx}$$

the perfectly general solution is

$$y = \frac{1}{(n-l)(n-m)} e^{nx} + Ae^{lx} + Be^{mx},$$

A, B being independent arbitrary constants, but if we attempt to apply this solution to the particular equation (1), we find, in the first place, that the coincidence of n with l and m renders the first term infinite and, in the second place, that the coincidence of m with l leaves us with only one effective constant, $A+B$. The method by which in the commoner text-books the passage from the general solution to that of a particular equation is made in such cases as this is unconvincing.

The fundamental problem is always the same: given a linear differential equation

$$G(y) = g(x, a_1, a_2, \ldots a_r) \quad \ldots \quad \ldots \quad (2)$$

of any order, $a_1, a_2, \ldots a_r$ being constants, $G(y)$ involving x in any way and being homogeneous in y and a finite number of its derivatives, and $g(x, a_1, a_2 \ldots a_r)$ being any function of $x, a_1, a_2, \ldots a_r$, and given a function $F(x, p_1, p_2, \ldots p_r)$ of x and of r parameters $p_1, p_2, \ldots p_r$ which is not infinite irrespective of x for the values $a_1, a_2, \ldots a_r$ of the parameters and is such that

$$G\{F(x, p_1, p_2, \ldots p_r)\} = h(p_1, p_2, \ldots p_r)\, f(x, p_1, p_2, \ldots p_r), \ \ldots \ (3)$$

where the right hand side is not identically zero for all values of $p_1, p_2, \ldots p_r$ and for the values $a_1, a_2, \ldots a_r$ of $p_1, p_2, \ldots p_r$ the function $f(x, p_1, p_2, \ldots p_r)$ takes the form $g(x, a_1, a_2, \ldots a_r)$; to construct as general a solution as possible of the equation (2).

Thus, in the original example, we know that

$$(D-1)^2 e^{px} = (p-1)^2 e^{px}, \quad \ldots \quad \ldots \quad (4)$$

145514
JL
Bm 21, N09 IM
N09.7

and we wish to utilise this knowledge in the formation of integrals of the particular equation (1).

It will be noticed that here the function $f(x, p)$, which is e^{px}, is actually the function $g(x, p)$; in practice this is usually the case, but it is quite irrelevant to the argument which follows, which enables us, for example, to deduce a solution of equation (1) from the more elaborate equation

$$(D-1)^2(xe^{px})=2(p-1)\left\{e^{px}+\tfrac{1}{2}(p-1)xe^{px}\right\}, \quad \ldots \quad (5)$$

taking for $f(x, p)$ the expression $\left\{e^{px}+\frac{1}{2}(p-1)xe^{px}\right\}$.

We consider first the construction of a particular integral of equation (2), equation (3) being assumed.

If $h(a_1, a_2,\ldots a_r)\neq 0$, we have from (3) and the known relation between the functions $f(x, p_1, p_2,\ldots p_r)$, $g(x, a_1, a_2,\ldots a_r)$, and because the left hand side is linear and homogeneous in y and its derivatives,

$$G\left\{F(x, a_1, a_2,\ldots a_r)/h(a_1, a_2,\ldots a_r)\right\}=g(x, a_1, a_2,\ldots a_r),$$

and $$y=F(x, a_1, a_2\ldots a_r)/h(a_1, a_2,\ldots a_r)$$

is a solution of (2). In any case, from (3), on account of the character of the operator G, we have, p_k being any one of the parameters $p_1, p_2,\ldots p_r$,

$$G\left\{\partial F/\partial p_k\right\}=(\partial h/\partial p_k)f(x, p_1, p_2\ldots p_r)+h(p_1, p_3,\ldots p_r)(\partial f/\partial p_k), \quad (6)$$

and precisely, when $h(a_1, a_2,\ldots a_r)$ is zero, we can derive from (6) the r equations

$$G\left\{\partial F/\partial a_k\right\}=(\partial h/\partial a_k)\, g(x, a_1, a_2,\ldots a_r),$$

for $k=1, 2,\ldots r$.

If then any of the functions $\partial h/\partial a_1$, $\partial h/\partial a_2,\ldots\partial h/\partial a_r$ are not zero, solutions of the original equation (2) are given by

$$y=\frac{\partial F}{\partial a_k}\bigg/\frac{\partial h}{\partial a_k}$$

for the corresponding values of k.

Thus from equation (5) we have

$$(D-1)^2\left\{\partial(xe^{px})/\partial p\right\}=2\left\{e^{px}+\tfrac{1}{2}(p-1)xe^{px}\right\}$$
$$+2(p-1)\left[\partial\left\{e^{px}+\tfrac{1}{2}(p-1)xe^{px}\right\}/\partial p\right], \quad \ldots \quad (7)$$

and from this it follows that

$$y=\tfrac{1}{2}\left[\partial(xe^{px})/\partial p\right]_{p=1}=\tfrac{1}{2}x^2e^x$$

is a solution of equation (1), the right hand side of equation (7) reducing, for $p=1$, to a constant multiple of e^x for the very reason that the factor $p-1$, which prevented our obtaining a solution of equation (1) by mere substitution in equation (5), remains a factor of that part of the expression which is not itself a constant multiple, for $p=1$, of e^x.

But it may happen that all the functions of the form $\partial h/\partial a_k$ are zero. For example, if we are using the simple equation (4) for the

purpose of solving equation (1), the function $h(p)$ of the only parameter p is $(p-1)^2$, and dh/dp as well as $h(p)$ is zero when p is equal to 1. And, in fact, differentiation of the equation (4) with respect to p gives us equation (5) itself, in the form

$$(D-1)^2\{\partial e^{px}/\partial p\}=2(p-1)e^{px}+(p-1)^2(\partial e^{px}/\partial p), \quad \ldots \quad (8)$$

and the right hand side of this equation does not reduce to a proper multiple of e^x for any value of p.

If however we differentiate the equation (6) with respect to any one of the parameters which it involves, we have for all values of the parameters and for all values of i, k from 1 to r.

$$G\left\{\frac{\partial^2 F}{\partial p_i\,\partial p_k}\right\}=\frac{\partial^2 h}{\partial p_i\,\partial p_k}f+\frac{\partial h}{\partial p_i}\frac{\partial f}{\partial p_k}+\frac{\partial h}{\partial p_k}\frac{\partial f}{\partial p_i}+h\frac{\partial^2 f}{\partial p_i\,\partial p_h}$$

and when h and its first derivatives with respect to the parameters p_i, p_k (which may coincide) vanish for the values $a_1, a_2, \ldots a_r$ of $p_1, p_2, \ldots p_r$, we have

$$G\{\partial^2 F/\partial a_i\,\partial a_k\}=(\partial^2 h/\partial a_i \partial a_k)g(x, a_1, a_2, \ldots a_r)$$

and if the second derivative $\partial^2 h/\partial p_i\,\partial p_k$ of h remains different from zero for the special values of the parameters, we can deduce a solution

$$y=\frac{\partial^2 F}{\partial a_i\,\partial a_k}\bigg/\frac{\partial^2 h}{\partial a_i\,\partial a_k}$$

of our original equation. For example, from equation (8) we have

$$(D-1)^2\{\partial^2 e^{px}/\partial p^2\}=2e^{px}+4(p-1)(\partial e^{px}/\partial p)+(p-1)^2(\partial^2 e^{px}/\partial p^2),$$

an equation which yields us a solution of the equation (1) in the form

$$y=\tfrac{1}{2}(\partial^2 e^{px}/\partial p^2)_{p=1}=\tfrac{1}{2}x^2e^x$$

simply *because* the earlier equations (4), (8) were ineffective, for this purpose.

The reasoning can be continued from step to step, and we conclude that if $h(p_1, p_2, \ldots p_r)$ and all those of its partial derivatives of the first $m-1$ orders which are subsidiary to a particular derivative

$$\partial^m h/\partial p_1{}^{s_1}\partial p^{s_2}\ldots\partial p_r{}^{s_r}$$

that is, which are of the form

$$\partial^n h/\partial p_1{}^{t_1}\partial p_2{}^{t_2}\ldots\partial p_r{}^{t_r} \text{ where } n<m,\ t_k \leqq s_k)$$

vanish when $p_1, p_2, \ldots p_r$ have the values $a_1, a_2, \ldots a_r$, but this derivative itself does not then vanish, a particular solution of equation (2) is given by

$$y=\frac{\partial^m F}{\partial a_1{}^{s_1}\partial a_2{}^{s_2}\ldots\partial a_r{}^{s_r}}\bigg/\frac{\partial^m h}{\partial a_1{}^{s_1}\partial a_2{}^{s_1}\ldots\partial a_r{}^{s_r}}$$

The common case is that in which there is only one parameter p, and only one constant a in the original equation, and the functions $f(x, p)$, $g(x, p)$ are the same ; h is a function of p alone, F is a function

of x and p, and the result takes the form that if h, dh/dp, d^2h/dp^2, ... $d^{m-1}h/dp^{m-1}$ all vanish for $p=a$, but d^mh/dp^m does not, a solution of the equation

$$G(y)=g(x, a)$$

is given by

$$y=\frac{\partial^m F}{\partial a^m}\Big/\frac{d^m h}{da^m},$$

$F(x, p)$ and $h(p)$ being such functions that

$$G\{F(x, p)\}=h(p)\cdot g(x, p).$$

In this case we obtain only one solution of the equation, but in the general case we often obtain a number of solutions, which may or may not be independent.

To see how a multiplicity of parameters may be utilised, consider the equation

$$(D-1)^2(D-2)y=e^x+e^{2x}, \qquad \dots \qquad \dots \qquad \dots \quad (9)$$

being given that

$$(D-1)^2(D-2)F(x, p, q)=h(p, q)\{e^{px}+e^{qx}\}$$

where

$$F(x, p, q)=(q-1)^2(q-2)e^{px}+(p-1)^2(p-2)e^{qx},$$
$$h(p, q)=(p-1)^2(p-2)(q-1)^2(q-2).$$

If we make both $p=1$ and $q=2$ in $h(p, q)$ or in any of its derivatives of the first three orders, except $\partial^3h/\partial p^2\partial q$, the result is zero, but this one derivative has the value -2. Hence one solution of equation (9) is given by

$$y=-\tfrac{1}{2}\frac{\partial^3F(x, 1, 2)}{\partial p^2\partial q}=-\tfrac{1}{2}x^2e^x+xe^{2x}.$$

It is easy to see that in this example no other integral is derivable from the functions $F(x, p, q)$, $h(p, q)$.

For a second example, in which we cannot, without the use of complex numbers, resolve the problem into two simpler ones, as we should do in practice in the last example, we may solve the equation

$$(D^2-2D+2)y=e^x \sin x, \quad \dots \qquad \dots \qquad \dots \quad (10)$$

premising that

$$(D^2-2D+2)F(x, p, q)=h(p, q)e^{px}\sin qx \quad \dots \qquad \dots \quad (11)$$

where

$$F(x, p, q)=e^{px}[\{(p-1)^2+(1-q^2)\}\sin qx-2(p-1)q\cos qx], \dots \quad (12)$$
$$h(p, q)=\{(p-1)^2+(1-q^2)\}^2+4(p-1)^2q^2. \qquad \dots \qquad \dots \quad (13)$$

We see, without difficulty that we may use either $\partial^2h/\partial p^2$ or $\partial^2h/\partial q^2$, and we find the two solutions

$$y=\tfrac{1}{4}e^x(\sin x-2x\cos x),$$
$$y=-\tfrac{1}{4}e^x(\sin x+2x\cos x),$$

each of which as a matter of fact includes a part usually absorbed into the complementary function.

Two comments are suggested by this last example. In the first place, we do not pretend to have here an exceptionally simple way of solving the equation (10); we are only concerned to shew that if the equation itself is contained under a more general form of which we have the solution, a solution of the equation is obtainable by a direct and unimpeachable process. And in the second place, instead of starting from a known result involving two parameters, we might have assumed that

$$(D^2-2D+2)\,F(x,\,p)=h(p)e^{px}\sin px, \quad \ldots \quad \ldots \ (14)$$

where

$$F(x,\,p)=e^{px}(\sin px+p\cos px), \quad \ldots \quad \ldots \ (15)$$

$$h(p)=-2(p-1)(p^2+1), \quad \ldots \quad \ldots \quad \ldots \ (16)$$

and obtained the slightly different solution

$$y=-\tfrac{1}{4}e^{x}(\cos x+2x\cos x).$$

A similar possibility is clearly open in general. All that is required of $p_1, p_2, \ldots p_r$ is that they should be capable of taking simultaneously the values $a_1, a_2, \ldots a_r$, and subject to this condition they may be made from the commencement any convenient functions of a single parameter p. Thus if $a_1, a_2, \ldots a_r$ are all different from zero, we may put $p_k=a_kp$, and the particular equation to be studied will correspond to $p=1$, or in any case, we may put $p_k=a_k+p$ in the general equation which is given to us and consider what happens when $p=0$. But, since different equations may most naturally be discussed in different ways, there is evident advantage in limiting ourselves as little as possible in the general treatment.

(*To be continued.*)

Some Properties of Cyclic Equations.

By M. T. NARANIENGAR.

1. Consider the equation

$$f(x) \equiv x^2+px+q=0. \qquad \dots \qquad \dots (1)$$

The roots of this will be α, α^2, where $\alpha^4 \equiv \alpha$, if equation (1) is included in $f(x^2)=0$. Hence $f(x)$ must be a factor of $f(x^2)$.

Changing the sign of x, we find that $f(-x)$ is also a factor of $f(x^2)$.

Thus we are led to the identity

$$f(x^2) \equiv f(x)\, f(-x). \qquad \dots \qquad \dots (2)$$

Suppose, now that z is a quantity such that $z^2=z$. ... ... (3)

Then substituting in (2) z for x and remembering that $z^2=z$, we have, since $f(z) \neq 0$

$$1=f(-z) \qquad \dots \qquad \dots \qquad \dots (4)$$

Equation (4) must be satisfied by the values of z found from (3).

Hence $\qquad z^2-pz+q-1 \equiv z^2-z.$

In other words $p=1$, $q=1$; that is, the roots of the cyclic equation

$$f(x) \equiv x^2+x+1=0$$

are α, α^2, where $\alpha^4 \equiv \alpha$.

2. Next, suppose the roots of (1) are α, α^3, where $\alpha^9 \equiv \alpha$. In this case, we find that equation (1) is included in $f(y)=0$, where $y=x^3$; whence we obtain the identity

$$f(x^3) \equiv f(x) \cdot f(xw),\ f(xw^2) \ \dots \qquad \dots \qquad \dots (5)$$

where w^3 is unity.

Equation (5) leads to

$$p=p^3-3pq\,;\ q=q^3\,;$$

whence $q=0,\ \pm 1$; $p=0,\ \pm 1,\ \pm 2,\ \pm\sqrt{-2}.$

Rejecting trivial and inadmissible values, we finally have the cylic quadratic equations

$$x^2+1=0,\ x^2 \pm x\sqrt{-2}-1=0, \text{ whose roots are } \alpha,\ \alpha^3\ (\alpha^8=1).$$

3. Similarly, if α, α^4 be the roots, where $\alpha^{16} \equiv \alpha$, we have the identity

$$f(x^4) \equiv f(x).\ f(xw).\ f(xw^2).\ f(xw^3), \qquad \dots \qquad \dots (6)$$

where $w^4=1$.

Equation (6) results in the following relations:

$$p=2q^2-(p^2-2q)^2,$$
$$q=q^4\,;$$

whence we easily obtain the quadratics $x^2+p_1x+1=0$, $x^2+p_2x+1=0$; where $p_1\ p_2$ are the roots of $p^2-p-1=0$.

In other words, $(x^2+p_1x+1)(x^2+p_2x+1)$ is a factor $(x^{15}-1)$. In fact, the product is $x^4+x^3+x^2+x+1=0$, so that the roots of the quadratics can be written in the form $\alpha, \alpha^4; \alpha^2, \alpha^8 (=\alpha^8)$.

4. Generally, let the roots of (1) be α, α^n where $\alpha^{n^2}\equiv\alpha$. Then from the identity

$$f(x^n)\equiv f(x).\ f(xw).\ f(xw^2).\ \ldots f(xw^{n-1}),\ \ldots \qquad \ldots (7)$$

where $w^n=1$, we deduce

$$p=(-1)^{n-1}\left\{p^n-n.\ p^{n-2}q+\frac{n.n-3}{1.2}.p\ p^{n-4}q^2-\right.$$
$$\left.\frac{n(n-4)(n-5)}{1.2.3}\ p^{n-6}q^3+-\ldots\right\}\ldots (8)$$

and

$$q=q^n.$$

Thus $q=\left(\cos\frac{2k\pi}{n-1}+i\sin\frac{2k\pi}{n-1}\right)$ for all k; and the values of p are found from equation (8).

Now, equation (8) is always reducible. For since α, α^n are the roots of $x^2+px+q=0$,

$$q=\alpha^{n+1};\ p=-(\alpha+\alpha^n)=-(\alpha+q\alpha^{-1}).$$

$$\therefore \qquad \alpha=q^{\frac{1}{n+1}}\left(\cos\frac{2k\pi}{n+1}+i\sin\frac{2k\pi}{n+1}\right),$$

and

$$\alpha^n=q\alpha^{-1}=q^{\frac{n}{n+1}}\left(\cos\frac{2kn\pi}{n+1}+i\sin\frac{2kn\pi}{n+1}\right)$$

$$=q^{\frac{1}{n+1}}\left(\cos\frac{2k\pi}{n+1}-i\sin\frac{2k\pi}{n+1}\right),$$

since $q^{n-1}=1$.

Hence

$$p=-2q^{\frac{1}{n+1}}\cos\frac{2k\pi}{n+1},$$

for all k, shewing that the values of p are the same for the pairs $(1, n)$. $(2, n-1)$, ... of values of k. In other words, the number of values of p cannot exceed $\frac{1}{2}(n+1)$.

Again, if we write $z=2\cos\frac{2k\pi}{n+1}$, we know that z satisfies an algebraic equation

$$\phi(z)=0,$$

which is of degree $\frac{1}{2}n$ or $\frac{1}{4}(n-2\pm1)$ according as n is even or odd,

Putting

$$p=-q^{\frac{1}{n+1}}z,$$

we deduce that equation (8) is reducible to

$$\phi\left(-p/q^{\frac{1}{n+1}}\right)=0,$$

which is of degree $\frac{1}{2}n$ or $\frac{1}{4}(n-2\pm1)$ according as n is even or odd.

The form of $\phi(z)$ may be obtained as follows:—

Suppose $\psi(m)\equiv z^m-m\,z^{m-2}+\frac{m(m-3)}{1\cdot2}z^{m-4}-\ldots\ldots;$

then

$$\phi(z)\equiv\psi\left(\frac{n}{2}\right)+\psi\left(\frac{n-2}{2}\right)+\ldots\ldots\ldots1=0,$$

if n is even;

and $$\phi(z)\equiv\psi\left(\frac{n-1}{4}\right)-\psi\left(\frac{n-5}{4}\right)+\ldots\ldots\ldots=0,$$

if n is odd and of the form $4m+1$; otherwise $\phi(z)\equiv\psi\left(\frac{n-1}{4}\right)$.

For, $\psi(m)=2\cos\frac{2mk\pi}{n+1}$, being the equivalent of $\lambda^m+\lambda^{-m}$, where $z=\lambda+\lambda^{-1}=2\cos\frac{2k\pi}{n+1}$; and

$$\phi(z)\equiv(\lambda^{\frac{1}{2}n}+\lambda^{-\frac{1}{2}n})+\lambda^{\frac{1}{2}(n-2)}+\lambda^{-\frac{1}{2}(n-2)}+\ldots\ldots$$

if n is even; and

$$\phi(z)=\lambda^m+\lambda^{-m}-\ldots\ldots\ldots$$

if n is $(4m+1)$, &c.

5. Consider, next, the cubic

$$f(x)\equiv x^3+px^2+qx+r=0. \quad \ldots \quad \ldots \quad (9)$$

If this should have roots α, α^2, α^4, where $\alpha^8=\alpha$, or $\alpha^7=1$, we write

$$f(x^2)\equiv-f(x).\ f(-x) \text{ as in § 1.} \quad \ldots \quad \ldots \quad (10)$$

Giving values 0, 1 to x, we find $r=-1$, $p-q=1$; and by equating co-efficients of x^4 in (10) $p^2+p-2q=0$. Thus, we have for p, the quadratic

$$p^2-p+2=0. \quad \ldots \quad \ldots \quad (11)$$

The equation for q is

$$q^2+q+2=0; \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

whose roots differ from the roots of the former only in sign.

Hence, remembering that $p-q=1$, we deduce the following relations:

(i) If $p_1\,p_2$ are the roots of $p^2-p+2=0$, the cubic equation $x^3+p_1x^2-p_2x-1=0$ is cyclic and has for roots α, α^2, α^4, where $(\alpha^7=1)$.

(ii) Also

$$(x^3+p_1x^2-p_2x-1)(x^3+p_2x^2-p_1x-1)\equiv(x^7-1)/(x-1), \quad \ldots \quad (13)$$

the second factor $(x^3+p_2x^2-p_1x-1)$ corresponding to the roots α^3, α^6., α^5 $(=\alpha^{12})$.

From the trigonometrical expression for α *viz* $\left(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}\right)$, we easily deduce that

$$\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)+i\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)=-p_1,$$

$$\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)-i\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}=\sin\frac{8\pi}{7}\right)=-p_2,$$

whence finally $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}=-\frac{1}{2}$, ... (14)

$$\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\tfrac{1}{2}\sqrt{7}, \quad \text{... ... (15)}$$

by virtue of equation (11).

6. If the roots of the cubic in the last article should be α, α^3, α^9 where $\alpha^{26}=1$, we obtain, as in § 2,

$$f(x^3)\equiv f(x).\ f(xw).\ f(xw^2). \quad [w^3=1].$$

Equating co-efficients

$$\left.\begin{aligned} p&=p^3-3pq+3r,\\ q&=q^3-3pqr+3r^2,\\ r&=r^3. \end{aligned}\right\} \quad \text{... ... (16)}$$

Hence if $r=1$, p, q are found from

$$p=p^3-3pq+3$$
$$q=q^3-3pq+3.$$

Rejecting the case $p=q$,

$$p^2+pq+q^2=1,\ p+q=p^3+q^3-6pq+6.$$

Put $p+q=a$, $pq=b$, then

$$a^2-b=1,\ a=a(1-2b)-6b+6.$$

Hence, eliminating b,

$$2a^3+6a^2-2a-12=0.$$

$\therefore$ $(a+2)(a^2+a-3)=0$ ... ... (17)

$\therefore$ $a=-2,\ b=3$; or $a=-\frac{1}{2}(1\pm\sqrt{13}),\ b=\frac{1}{2}(5\pm\sqrt{13})$.

The first set of values determines p, q as the roots of the quadratic $\lambda^2+2\lambda+3=0$; and the corresponding cubic x^3+px^2+qx+1 is divisible by $x-1$, since the sum of the co-efficients is $p+q+2=0$. In this case, therefore, the property investigated relates to the resulting quadratic $x^2+(1+\lambda)x-1=0$, and is readily seen to be identical with that stated in § 2.

Taking a as a root of $a^2+a-3=0$ and $b=a^2-1=2-a$, we have the following :—

If λ, λ_2 are the roots of $\lambda^2-a\lambda+b=0$, the equation $x^3+\lambda_1 x^2+\lambda_2 x+1=0$ is cyclic and will have for roots α, α^3, α^9. $[\alpha^{13}=-1]$.

Similarly, if $r=-1$, we find that the cubic

$$x^3-\lambda_1 x^2+\lambda_2 x-1=0$$

will have roots α, α^3, α^9, where $\alpha^{13}=1$.

As in § 5, writing $\alpha=\cos\frac{2\pi}{13}+i\sin\frac{2\pi}{13}$, we deduce

$$\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{18\pi}{13}=\tfrac{1}{2}a, \quad \ldots \quad \ldots (18)$$

$$\sin\frac{2\pi}{13}+\sin\frac{6\pi}{13}+\sin\frac{18\pi}{13}=\tfrac{1}{2}\sqrt{(5-3a)}. \quad \ldots (19)$$

7. In the examples considered in §§ 5 and 6, we might have proceeded as follows :—

Suppose the cubic is $x^3-px^2+qx-r=0$, and that the roots are α, α^2, α^4, $(\alpha^7=1)$.

Then we have

$$p=\alpha+\alpha^2+\alpha^4$$
$$q=\alpha.\alpha^2+\alpha^2.\alpha^4+\alpha^4.\alpha$$
$$=\alpha^3+\alpha^6+\alpha^5,$$
$$r=\alpha.\alpha^2.\alpha^4=\alpha^7=1.$$

$\therefore$ $p+q=\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6=-1$, as is well-known;

and $p.q=(\alpha+\alpha^2+\alpha^4)(\alpha^3+\alpha^6+\alpha^5)$

$$=\alpha^4+\alpha^7+\alpha^6+\alpha^5+\alpha^8+\alpha^7+\alpha^{10}+\alpha^9$$
$$=3+q+p, \text{ since } \alpha^7=1.$$

Hence p, q are the roots of the quadratic equation

$$\lambda^2+\lambda+2=0.$$

In other words, if λ_1, λ_2 are the roots of this quadratic, the roots of the cubic

$$x^3-\lambda_1 x^2+\lambda^2 x-1=0$$

are α, α^2, α^4, where $\alpha^7=1$.

Again, let the roots be α, α^3, λ^9. $[\alpha^{13}=1]$. Then

$$p=\alpha+\alpha^3+\alpha^9$$
$$q=\alpha.\alpha^3+\alpha^3.\alpha^9+\alpha^9\alpha$$
$$=\alpha^4+\alpha^{12}+\alpha^{10}$$
$$r=\alpha.\alpha^3.\alpha^9=\alpha^{13}=1.$$

$\therefore$ $p^2=\alpha^2+\alpha^6+\alpha^5+2(\alpha^4+\alpha^{12}+\alpha^{10})$

$$q^2=\alpha^8+\alpha^{11}+\alpha^7+2(\alpha^3+\alpha^9+\alpha)$$

$\therefore$ $$p^2+q^2=-1+p+q, \text{ since } \sum_{k=1}^{12}(\alpha^k)=-1;$$

and $$pq=\alpha^5+1+\alpha^{11}+\alpha^7+\alpha^2+1+1+\alpha^8+\alpha^6$$
$$=3-1-p-q.=2-p-q.$$

$\therefore$ $$p^2+q^2+pq=1.$$

Putting $p+q=a$, $pq=b$, we find
$$a^2-b=1;\ a+b=2.$$

$\therefore$ $$a^2+a-3=0.$$

8. We shall now consider the general irreducible cyclic equation of the third degree, *viz.*
$$f(x)\equiv x^3+px^2+qx+r=0.$$

Let the roots α_1 α_2 α_3 of this equation be connected by the relations
$$\alpha_2=\phi(\alpha_1), \alpha_3=\phi(\alpha_2),\ \alpha_1=\phi(\alpha_3).$$

By the Tschirnhausen transformation we know that the most general form for ϕ is of the second degree, so that we may write
$$\phi(\alpha)\equiv a\alpha^2+b\alpha+c=a(x-h)(x-k) \text{ say.} \quad \dots (20)$$

Hence $f(x)$ must be a factor of $f[ax^2+bx+c]$.

Replacing x by $(h+k-x)$, we see that
$$ax^2+bx+c \text{ or } a(x-h)(x-k)$$
remains unchanged, and therefore
$$f(-b/a-x) \text{ or } f(h+k-x) \text{ is also a factor of } f[ax^2+bx+c].$$

Thus we have the identity
$$f[ax^2+bx+c]\equiv -a^3f(x).\ f(-b/a-x). \quad \dots \quad \dots (21)$$

Putting $x=\lambda$, μ in (21) where λ, μ are the roots of
$$ax^2+bx+c=x, \quad \dots \quad \dots \quad \dots (22)$$
we get two linear relations among the co-efficients p, q, r; *viz*:
$$1=-a^3f(-b/a-\lambda),$$
$$1=-a^3f(-b/a-\mu).$$

In fact, the expression $(ax^2+bx+c-x)$ is a factor of
$$\{1+a^3.\ f(-b/a-x)\}.$$

Equating co-efficients of x^4 in (21) a third relation is obtained whence the values of p, q, r easily follow.

[Cajori's *Theory of Equations*, § 176.]

9. Question 507 of Mr. S. Ramanujan is an example of the above method. [J. I. M. S., Vol. VI, No. 2, p. 74.]

To solve the equations $x^2=y+a$, $y^2=z+a$, $z^2=x+a$; we write
$$f(x)\equiv x^3+px^2+qx+r,$$
and deduce, as in § 8,
$$f(x^2-a)\equiv -f(x).\ f(-x) \dots \quad \dots \quad \dots$$

The roots of

$$x^2-x-a=0 \qquad \dots \qquad \dots \qquad \dots \qquad (24)$$

are $\frac{1}{2}(1\pm\sqrt{1+4a})$. Call them λ, μ; so that

$$\lambda+\mu=1,\ \lambda\mu=-a. \quad \dots \qquad \dots \qquad \dots \qquad (25)$$

Put $x=\lambda$, μ in (23), and we have

$$\left.\begin{aligned} 1&=\lambda^3-p\lambda^2+q\lambda-r,\\ 1&=\mu^3-p\mu^2+q\mu-r. \end{aligned}\right\} \qquad \dots \qquad \dots \qquad (26)$$

$\therefore$ $$0=\lambda^2+\lambda\mu+\mu^2-p\,(\lambda+\mu)+q$$

i.e., $$0=1+a-p+q, \text{ using (25)} \qquad \dots \qquad \dots \qquad (27)$$

Also, eliminating q from (26)

$$a-1-pa-r=0. \qquad \dots \qquad \dots \qquad \dots \qquad (28)$$

The third equation connecting p, q, r, is got from the constant term of (23), *viz*:

$$r^2+r-a^3+pa^2-qa=0. \qquad \dots \qquad \dots \qquad \dots \qquad (29)$$

Equations (27), (28), (29) will give two sets of values for p, q, r corresponding to the two cubic factors of the sextic

$$[\{(x^2-a)^2-a\}^2-x-a]\div(x^2-x-a)$$

10. As a last example, we shall investigate a cyclic cubic equation whose roots are connected by the following relations:

$$\alpha_2=\alpha_1^2-2\alpha_1,\ \alpha_3^2=\alpha_2^2-2\alpha_2,\ \alpha_1^2=\alpha_3^2-2\alpha_3.$$

Our method leads to the identity

$$f(x^2-2x)\equiv-f(x).\ f(2-x). \qquad \dots \qquad \dots \qquad (30)$$

Putting $x=0$, 3, we deduce

$$1=-(8+4p+2q+r)$$
$$1=-(-1+p-q+r).$$

$\therefore$ $$p+q+3=0 \qquad \dots \qquad \dots \qquad \dots \qquad (31)$$

Also, equating co-efficients of x^4 in (30)

$$12+p=12+4p+q+p(-6-p)+q.$$

$\therefore$ $$p^2+3p+2q=0. \qquad \dots \qquad \dots \qquad (32)$$

From (31) and (32)

$$p^2+5p+6=0,$$

or $$(p+2)(p+3)=0.$$

Hence

$$p=-2,\ q=-1,\ r=1;$$
$$p=-3,\ q=0,\ r=3.$$

In other words, the cubic equations

$$x^3-2x^2-x+1=0,$$
$$x^3-3x^2+3=0,$$

are cyclic, the relations connecting the roots being

$$\alpha_2=\alpha_1^2-2\alpha_1,\ \alpha_3^2=\alpha_2^2-2\alpha_2,\ \alpha_1^2=\alpha_3^2-2\alpha_3.$$

SHORT NOTES.

A note on $\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}dx$.

The value of this integral is usually given as $\frac{\pi}{b}\log\frac{a+b}{b}$, but this result is affected by a change in the sign of a or b, whereas the integrand is independent of the sign of a or b. The present note is intended to explain this apparent fallacy.

Let $$I=\int_{-\infty}^{+\infty}\frac{\log(1+a^2x^2)}{1+b^2x^2}dx;$$

so that $$\frac{\partial I}{\partial a}=2a\int_{-\infty}^{+\infty}\frac{x^2}{(1+a^2x^2)(1+b^2x^2)}dx.$$

This can now be integrated by the methods of partial fractions.

But the correct result is obtained by integrating round a suitable contour.

The function $\frac{z^2}{(1+a^2z^2)(1+b^2z^2)}$ has four simple poles, *viz*:

$$z=\pm\frac{i}{b},\ z=\pm\frac{i}{b}.$$

Take a contour on the upper half of the z—plane consisting of the real axis from $-\infty$ to $+\infty$, and a "large" semi-circle Γ.

(1) If a and b are both positive, the only poles of the function within the contour are $+\frac{i}{a}$, $+\frac{i}{t}$.

(2) If a and b are both negative, then the poles of the function within the contour are $-\frac{i}{a}, -\frac{i}{b}$.

In any case the two poles on the upper half of the z—plane are

$$\frac{i}{a_1}\ \frac{i}{b_1}$$

and $I=2\pi i\times$ sum of the residues at the poles $\frac{i}{a_1}, \frac{i}{b_1}$,

where a_1, b_1, the numerical values of for a, b.

By the usual method we find

$$\text{sum of the residues}=\frac{1}{i\,a_1\,b_1\,(a_1+b_1)}.$$

Now $\int_C=\int_\Gamma+\int_{-\infty}^{+\infty}$. At all points on Γ the integrand is sensibly equal to $\frac{1}{a^2b^2z^2}$ which vanishes in the limit when the radius of Γ is sufficiently large.

$$\therefore \qquad \frac{\partial I}{\partial a} = \int_{\infty}^{\infty} \frac{x^2}{(1+a^2x^2)(1+b^2x^2)} dx = 2\pi i \times \frac{1}{i\, a_1 b_1 (a_1+b_1)}.$$

$$\therefore \qquad I = \frac{2\pi}{b_1} \log \frac{a_1+b_1}{b^1}.$$

and the required integral is $\frac{\pi}{b_1} \log \frac{a_1+b_1}{b_1}$,

which is the correct form of the result.

The results

$$\int_0^{\frac{1}{2}\pi} \log (\alpha^2 \cos^2\theta + \beta^2 \sin^2\theta)\, d\theta = \pi \log \tfrac{1}{2} (\alpha+\beta)$$

and

$$\int_0^{\pi} \frac{\cos 2\, rx}{1-2a \cos x + a^2} = \frac{\pi a^{2r}}{1-a^2}$$

can also be similarly explained

The discussion given here is probably nothing new, but the fact that it is not explained in some of the well-known text-books on Calculus is perhaps a sufficient justification for this note.

J. M. Bose.

The Face of the Sky for March and April 1915.

Sidereal time at 8 p.m.

	March.					*April.*		
D.	H.	M.	S.			H.	M.	S.
1	6	33	14	...	...	8	35	27
8	7	0	49	...	...	9	3	3
15	7	28	25	...	...	9	30	39
22	7	55	1	...	...	9	58	14
29	8	22	37	...	...	50	25	50

From this table the constellations visible during the evenings can be ascertained by a reference to their positions as given in the star-chart,

Phases of the Moon.

	March.				April.			
	D.	H.	M.		D.	H.	M.	
Full Moon ...	2	0	3	A.M.	...			
Last Quarter ...	8	5	58	P.M.	... 7	1	42	A.M.
New Moon ...	16	1	12	A.M.	... 14	5	6	P.M.
First Quarter ...	24	4	18	„	... 22	9	9	„
Full Moon ...	31	11	8	„	... 29	7	49	„

It may be noticed that January and March have each two Full Moons and February is without a Full Moon.

The Sun

enters the Vernal Equinox on March 21 at 10-21 A.M.

The Planets.

Mercury is stationary on March 6 and attains its greatest elongation (27° 43′ W) on March 20. It is in conjunction with the Moon on March 13 and April 13, with Jupiter on March 30 and with Mars on April 4.

Venus is in conjunction with the Moon on March 12 and April 11, with Uranus on March 19 and with Jupiter on April 15. It is near the boundary between Capricornus and Aquarius in March and in Pisces in April.

Mars is in conjunction with the Moon on March 14 and April 12 and with Jupiter on March 24. It is in Pisces in March and in Aries in April.

Jupiter is in conjunction with the Moon on March 14 and April 11. It is in Pisces in these months.

Saturn is in quadrature to the Sun on March 17. It is in conjunction with the Moon on March 23 and April 20.

Uranus is in conjunction with the Moon on March 12 and April 9.

Neptune is in quadrature to the Sun on April 19 and is in conjunction with the Moon on March 26 and April 22.

V. RAMESAM.

SOLUTIONS.

Question 548.

(**A. Narasinga Rao**):—If $u_r = \dfrac{x^r}{1-x^r}$, shew that the infinite series

$$u_1+u_2+u_3+u_5-u_6+u_7-u_{10}\ldots\ldots = x\,\frac{1+x}{1-x},$$

where the suffixes of u are the numbers having non-repeated prime factors, the terms being taken with a $+$ *ve* or $-$ *ve* sign according as the number of such factors is odd or even.

Solution by T. P. Trivedi, M.A., LL.B., N. Durairajan, R. Vythynathaswamy and N. Sankara Aiyar.

Let us find the coefficient of x^N by picking out the terms in which it occurs.

Let $N = a^\alpha b^\beta c^\gamma\ldots\ldots$, where a, b, c are the prime factors of N, which are n in number.

Then x^N occurs in u_1 which is $1+x+x^2\ldots\ldots\ x^N+\ldots+\ldots$

It occurs also

once with the $+$ sign in each of the terms ua, ub, uc, ...,

once with the $-$ sign in each of $-uab$, $-uac$, $-uad$, ...,

once with the $+$sign in each of $uabc$, $uabd$, ...,

and so on. Hence the coefficient of x^N

$$= 1+n-\tfrac{1}{2}\,n\,(n-1)+\frac{1}{3\,!}\,n\,(n-1)\,(n-2)-\ldots$$

$$= 2 - \left\{1-n+\frac{n\,(n-1)}{1\cdot 2}-\frac{n\,(n-1)\,(n-2)}{1\cdot 2\cdot 3}+\ldots\right\}$$

$$= 2-(1-1)^n$$

$$= 2.$$

The coefficient of x is 1.

Thus the given series $=x+2x^2+2x^3+\ldots$

$$=x+\frac{2x^2}{1-x}=\frac{x+x^2}{1-x}$$

$$= x\,\frac{1+x}{1-x}.$$

Question 550.

(K. J. SANJANA) :—At a game of bridge recently played it was found that every player had three cards of each of three suits and four cards of the remaining suit; what is the chance of this distribution of the cards?

Solution by T. P. Trivedi, M. A., LL.B., and A. Narasinga Rao.

It is obvious that the 52 cards can be distributed amongst all the four players in the most general manner in $\frac{\underline{|52}}{\{\underline{|13}\}^4}$.

According to the data in the question, the first player may get his cards in ${}^4C_1\times{}^{13}C_3\times{}^{13}C_3\times{}^{13}C_3\times{}^{13}C_4$, ways, since he may have 4 cards of any of the four suits; the second can now get his cards in ${}^3C_1\times{}^{10}C_3\times{}^{10}C_3\times{}^9C_3\times{}^{10}C_4$ ways; the third player may have his cards in ${}^2C_1\times{}^7C_3\times{}^6C_3+{}^6C_3\times{}^7C_4$ ways; the rest of the cards go to the last player. The number of ways of distributing the cards is thus

$$\frac{\underline{|4}\,(\underline{|13})^4}{(\underline{|3})^{12}(\underline{|4})^4}, \text{ by multiplying the above result.}$$

Hence the chance of this distribution is

$$\frac{(\underline{|13})^8}{(\underline{|3})^{12}(\underline{|4})^8\,\underline{|52}}.$$

Question 552.

(S. KRISHNASWAMI AIYANGAR) :—Find the sum of

$$\frac{2}{3}x+\frac{1}{3\cdot5\cdot2}\,\frac{x^2}{1!}+\frac{1\cdot3}{5.7\cdot2^3}\,\frac{x^3}{2!}+\frac{1\cdot3\cdot5}{7\cdot9\cdot2^5}\cdot\frac{x^4}{3!}+\ldots\ldots$$
$$+\frac{1\cdot3\cdot5\ldots(2n-1)}{(2n+1)(2n+3)}\cdot\frac{1}{2^{2n-1}}\cdot\frac{x^{n+1}}{n!}+\ldots\ldots$$

Solution by T. P. Trivedi, M.A., LL. B. and K. J. Sanjana, M.A.

The general term is

$$\frac{1\cdot3\cdot5\cdot\ldots\ldots(2n-1)}{(2n+1)(2n+3)}\cdot\frac{1}{2^{2n-1}}\cdot\frac{x^{n+1}}{n!}$$

$$=\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots\ldots2n}\cdot\frac{1}{2n+1}\cdot\frac{x^{n+\frac{3}{2}}}{2n+3}\cdot\frac{1}{x^{\frac{1}{2}}2^{n-1}}$$

$$=\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\,\frac{2}{2n+3}\cdot\frac{x^{n+\frac{3}{2}}}{2^{n+\frac{1}{2}}}\cdot\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\cdot\frac{1}{2n+1}$$

$$=\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\cdot\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\int_0^x\frac{x^{n+\frac{1}{2}}}{2^{n+\frac{1}{2}}}dx\cdot\frac{1}{2n+1}$$

Now, since $\sin^{-1}z=\Sigma\,\frac{1\cdot3\cdot5\ldots2n-1}{2\cdot4\cdot6\ldots2n}\,\frac{z^{2n+1}}{2n+1}$, the summation being confined to odd powers of z, the above expression is equal to

$$\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\int_0^x\sin^{-1}\sqrt{\frac{x}{2}}\,dx$$

$$=\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\left\{x\sin^{-1}\sqrt{\frac{x}{2}}-\int\frac{x}{\sqrt{1-\frac{x}{2}}}\cdot\frac{1}{2\sqrt{2x}}dx\right\}$$

$$=\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\left\{x\sin^{-1}\sqrt{\frac{x}{2}}-\frac{1}{2}\int\frac{x\,dx}{\sqrt{2x-x^2}}\right\}$$

$$=\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\left\{x\sin^{-1}\sqrt{\frac{x}{2}}+\tfrac{1}{2}\sqrt{2x-x^2}-\tfrac{1}{2}\operatorname{vers}^{-1}x\right\},$$

no constant being added, since the expression as well as the integral vanish with x.

Question 553.

(S. Krishnaswami Aiyangar) :—ABC is a triangle such that its sides subtend equal angles at one focus S of an inscribed conic. With S as focus four more conics are drawn circumscribing the triangle. Show that

(i) The points of contact with the sides of the inscribed conic are on the focal radii through the vertices of the triangle.

(ii) The directrix of one of the circumconics is identical with the directrix of the inscribed conic corresponding to S.

(iii) The directrix of one of the other circumconics and its tangent through C meet at the point of intersection of AB and the directrix of the in-conic.

(iv) If this point is P_3, then P_3SC' is a right angle, where C′ is the point of contact of the inscribed conic with the side AB.

Solution by R. J. Pocock and R. Vythynathaswamy.

Since tangents, subtend equal angles at the focus and AB, BC, CA also subtend equal angles at S, it follows that the angles BSC′, C′SA

ASB′ are all equal to 60° and therefore BSB′ is a straight line, which proves (i).

Let the bisector of ∠BSA′ meet AB in P_3. Then $P_3SC' = 90°$ which proves (4), since P_3 will now be seen to be identical with P_3 of the question.

If CC′ cut the inscribed conic in F, P_3F is a tangent and since FC′ is a focal chord, P_3 lies on the directrix of the inscribed conic; similarly for P_2.

But since P_3S and P_2S bisect BSA′, and BSC′, P_2, P_3 lie on the directrix of the circumconic, which proves (ii).

Also since P_3S bisects ∠BSA′ and $\angle P_3SC = 90°$, P_3 lies on the directrix of the circum-conic. Thus P_3C is tangent at C, which proves (iii).

Question 554.

(T. P. Trivedi, M. A., LL. B.):—If V denote the entire volume of the solid $\left(\frac{x}{a}\right)^n + \left(\frac{y}{2}\right)^n + \left(\frac{z}{c}\right)^n = 1$, and A the whole area of its trace on the xy-plane, prove that

$$\frac{V}{A} = \frac{4c}{3n}. \quad B\left(\frac{2}{n}, \frac{1}{n}\right).$$

and deduce that for the solid

$$\left(\frac{x}{a}\right)^{\frac{2p}{2p+1}} + \left(\frac{y}{b}\right)^{\frac{2p}{2p+1}} + \left(\frac{z}{c}\right)^{\frac{2p}{2p+1}} = 1$$

$$\frac{V}{A} = \frac{4c}{9p} \cdot \frac{(p+1)(2p+1)}{(2p+3)(4p+3)}. \quad B\left(\frac{1}{2p}, \frac{1}{p}\right).$$

Solution by R. J. Pocock, B.A., B. Sc., F. R. A. S. and R. Vythynathaswamy.

As V, A are both Dirichlet Integrals, by the ordinary theorems

$$V = \frac{8abc}{n^3} \cdot \frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^3}{\Gamma\left(\frac{3}{n}\right)} \cdot \int_0^1 x^{\frac{3}{n}-1}\, dx = \frac{8abc}{3n^2} \frac{\left\{\Gamma\left(\frac{1}{n}\right)\right\}}{\Gamma\left(\frac{3}{n}\right)}$$

$$A = \frac{4ab}{n^2} \cdot \frac{\left\{\Gamma\left(\frac{1}{n}\right)\right\}^2}{\Gamma\left(\frac{2}{n}\right)} \cdot \int_0^1 \left\{x^{\frac{2}{n}-1}\, dx\right\} = \frac{4ab}{2n} \frac{\left\{\Gamma\left(\frac{1}{n}\right)\right\}}{\Gamma\left(\frac{2}{n}\right)};$$

$$\therefore \frac{V}{A} = \frac{4c}{3n} \cdot \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)}{\Gamma\left(\frac{3}{n}\right)} = \frac{4c}{3n}. \quad B\left(\frac{2}{n}, \frac{1}{n}\right).$$

Put $n=\frac{2p}{2p+1}$,

$$\therefore \qquad \frac{V}{A}=\frac{4c}{6p}(2p+1).\frac{\Gamma\left(1+\frac{1}{2p}\right)\Gamma\left(+\frac{3}{2}\frac{2}{2p}\right)}{\Gamma\left(3+\frac{3}{2p}\right)}$$

$$=\frac{8c}{9p}.\frac{(p+1)(2p+1)}{(2p+3)(4p+3)}.\quad B\left(\frac{1}{p},\frac{1}{2p}\right).$$

Question 555.

(R. VYTHYNATHASWAMY) :—P is a point on one of a system of curves $f(r_1, r_2, \alpha)=0$ where α is a variable parameter. If dn is the distance at the point P between this curve and its consecutive whose parameter is $\alpha+d\alpha$, show that

$$\frac{dn}{d\alpha}=\frac{1}{PQ}.\frac{\partial f}{\partial \alpha},$$

PQ being the diagonal of a parallelogram whose adjacent sides are along r_1, r_2, and equal to $\frac{\partial f}{\partial r_1}$ and $\frac{\partial f}{\partial r_2}$ respectively.

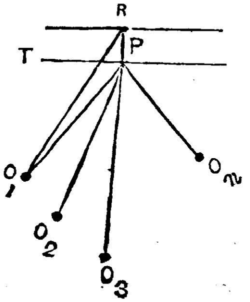
Solution (1) by N. Durairajan; (2) by T. P. Bhaskarasastri M. K. Kevalramani and A. Narasinga Rao.

(1) The theorem may be proved for the general case as follows :—

Let PT be the tangent at P to the curve $f(r_1 r_2 \ldots r_n, \alpha)=0$; and let the $\angle$T $PO_n=\alpha_n$.

Differentiating with respect to the arc, we have

$$\frac{\partial f}{\partial r_1}.\frac{dr_1}{ds}+\cdots\frac{\partial f}{\partial r_n}.\frac{dr_n}{ds}=0.$$



i.e. $f_1 \cos \alpha_1 + \ldots f_n \cos \alpha_n = 0,$ ... ... (i)
where f_m denotes $\partial f/\delta r_m$.

Now, the distance between a curve and its consecutive is the common normal to the two at any selected point on the first.

Hence, if we measure a distance dn equal to PR on the normal at P outwards, R will be the point on the consecutive curve.

The distances of R from the poles are

$$r_1 + dn \,.\, \sin \alpha_1, \; r_2 + dn \,.\, \sin \alpha_2, \text{ etc.}$$

which must satisfy the equation $f(r_1\, r_2 \ldots\ldots r_n,\, \alpha + d\alpha) = 0$.

$$\therefore \quad f(r_1 + dn \sin \alpha_1 \ldots\ldots\ldots\ldots, \alpha + d\alpha) = 0.$$

$$\therefore \quad f(r_1 \ldots\ldots \alpha) + \frac{\partial f}{\partial r_1}.\, dn \sin \alpha_1 + \ldots\ldots + \frac{\partial f}{\partial \alpha}.\, d\alpha) + \ldots\ldots = 0.$$

$$\therefore \quad \frac{\partial f}{\partial r_1}.\, dn \sin \alpha_1 + \ldots\ldots \frac{\partial f}{\partial \alpha}.\, d\alpha = 0,$$

since $f(r_1, r_2 \ldots\ldots \alpha) = 0$.

Thus $$f_1 \sin \alpha_1 + \ldots\ldots\ldots\ldots = -\frac{d\alpha}{dn}.\frac{\partial f}{\partial \alpha}. \quad \ldots \quad \text{(ii)}$$

Squaring and adding the relations (i) and (ii), we get

$$(f_1 \cos \alpha + \ldots)^2 + (f_1 \sin \alpha_1 + \ldots)^2 = \left(\frac{d\alpha}{dn}\right)^2 \left(\frac{\partial f}{\partial \alpha}\right)^2.$$

$$i.e., \quad f_1^2 + \ldots f_n^2 + \sum 2.\, f_1 f_2.\, \cos(\alpha_1 - \alpha_2) = \left(\frac{d\alpha}{dn}\right)^2 \left(\frac{\partial f}{\partial \alpha}\right)^2.$$

Also, if PQ is the resultant of forces $f_1 f_2 \ldots f_n$ at P along $r_1, r_2 \ldots r_n$,

$$PQ^2 = \Sigma(f_1^2) + 2\Sigma f_1 f_2 \cos(\alpha_1 - \alpha_2).$$

Hence $$\frac{dn}{d\alpha} = \frac{1}{PQ} \; \frac{\partial f}{\partial \alpha}.$$

(2) It is proved in Williamson, *Diff. Calculus*, p. 234, § 193, that PQ is normal to the curve at P.

Let r_1, r_2 be the bipolar coordinates of P, and $r_1 + dr_1, r_2 + dr_2$, those of R, where the normal cuts the consecutive curve. Then $PR = dn$.

Now dr_1 = projection of dn on radius vector $= -dn.\, r_1 \dfrac{d\theta_1}{ds}$.

Similarly $$dr_2 = -dn.\, r_2.\frac{d\theta}{ds}.$$

Since R is a point on the consecutive curve whose parameter is $\alpha + d\alpha$, we have

$$\frac{\partial f}{\partial r_1}.\, dr_1 + \frac{\partial f}{dr_2}.\, dr_2 + \frac{\partial f}{\partial \alpha} d\alpha = 0.$$

$$\therefore \quad dn \left(r_1 \frac{\partial f}{\partial r_1}.\, \frac{d\theta_1}{ds} + r_2 \frac{\partial f}{\partial r_2}.\, \frac{d\theta_2}{ds} \right) = \frac{\partial f}{\partial \alpha}.\, d\alpha.$$

But $$PQ = r_1 \frac{\partial f}{\partial r_1} \cdot \frac{d\theta_1}{ds} + r_2 \frac{\partial f}{\partial r_2} \cdot \frac{d\theta_2}{ds}$$

$\therefore$ $$PQ.\ dn = \frac{\partial f}{\partial \alpha} \cdot d\alpha.$$

$\therefore$ $$\frac{dn}{d\alpha} = \frac{1}{PQ} \cdot \frac{\partial f}{\partial \alpha}.$$

Question 556.

(R. VYTHYNATHASAWMY) :—Shew that in a cissoid the product of the perpendicular from the cusp on a tangent and the length of the tangent between the point of contact and the asymptote varies as the ordinate of the point of contact.

Solution by A. V. K. Krishna Menon, M.A., B.L., L.T. and several others.

The equation of the cissoid is $y^2(2a-x)=x^3$.

Let (x, y) be the point of contact.

If ψ be the angle the tangent make with the x axis and t the length of the tangent between the point of contact and the asymptote, we have

$$t=(2a-x)\sec\psi.$$

Also, if p be the perpendicular from the cusp on the tangent

$$p=\left(x\frac{dy}{dx}-y\right)\cos\psi.$$

Now $$y\frac{dy}{dx}=\frac{x^2(3a-x)}{(2a-x)^2}$$

$\therefore$ $$xy\frac{dy}{dx}-y^2=\frac{x^3(3a-x)}{(2a-x)^2}-\frac{x^3}{2a-x}=\frac{ax^3}{(2a-x)^2}$$

$\therefore$ $$tpy=(2a-x)\left(xy\frac{dy}{dx}-y^2\right)=(2a-x)\frac{ax^3}{(2a-x)^2}$$

$$=\frac{ax^3}{2a-x}=ay^2$$

Hence $$tp=ay,$$

or $$tp \propto y.$$

Question 557.

(PROFESSOR SANJANA) :—At a point P of a parabola the norma chord PQ is drawn meeting the axis in G and PN is the ordinate to the axis, prove that PQ. AN $=2$ SP. PG.

If from P two chords PP_1, PP_2 are drawn normal to the curve at P_1, P_2, prove further that

$SP_1.SP_2=AS.SP$, $PP_1.PP_2=PQ.AN$, where S is the focus.

Shew also that the circle P_1P_2S passes through the circumcentre of PP_1P_2.

Solution by A. V. K. Krishna Menon, M.A., B.L., L.T.; R. Vythynathaswamy, V. D. Gokhale, V. V. S. Narayanan, D. Krishnamurti and L. N. Subramaniam, M.A., L.T.

(i) Let P be the point $(at^2, 2at.)$ The equation of PQ is $tx+y=at^3+2at$. Where this meets the parabola $y^2t+4ay=4a^2t^3+8a^2t$.

i.e. $(y-2at)\,[yt+(2at^2+4a)]=0$.

$\therefore$ If y', y'' be the ordinates of P and Q,

$$\frac{PG}{QG}=\frac{y'}{y''}=\frac{2at}{(2at^2+4)/t}=\frac{2at^2}{2at^2+4a}$$

$$=\frac{at^2}{at^2+2a}=\frac{AN}{AG},\text{ with the usual notation.}$$

$$\therefore\quad \frac{PG}{PQ}=\frac{AN}{AN+AG}=\frac{AN}{2(a+AN)}=\frac{AN}{2SP}$$

$$\therefore\quad PQ.\ AN=2SP.\ PQ.$$

This proves the first part.

(ii) Now equation of the curve passing through the feet of the normals drawn from P $(at^2, 2at)$ is

$$(at^2-x)y+(2at-y)2a=0.$$

Where this meets theparabola

$$(4at^2t^2-4y^2)y+(8a^2t-4ay)2a=0$$

ie. $$y^3+4a^2y(2-t^2)-16a^3t=0$$

ie. $$(y-2at)(y^2+2aty+8a^2)=0.$$

If h_1 and y_2 are the ordinates of P_1 and P_2, they are the roots of the equation $y^2+2aty+8a^2=0$.

If x_1 and x_2 are the abscissæ of P_1 and P_2

$$SP_1.\ SP_2=(a+x_1)(a+x_2)$$

$$=a^2+a(x_1+x_2)+x_1x_2$$

$$=a^2+\tfrac{1}{4}(y_1^2+y_2^2)+\frac{1}{16a^2}y_1^2y_2^2$$

$$=a^2+\tfrac{1}{4}[4a^2t^2-16a^2]+4a^2,\text{ since } y_1+y_2=2at \text{ and } y_1y_2=8a^2,$$

$$=a(at^2+a)=a(AN+a)=AS.\ SP.$$

Again, if PP_1 and PP_2 cut the axis in G_1 and G_2

$PP_1.\ x_1=2SP_1.\ PG_1$, and $PP_2.\ x_2=2SP_2.\ P_2G_2$ [using the 1st part].

$\therefore$ $PP_1.\ PP_2.\ x_1x_2=4SP_1.\ SP_2.\ P_1G_1.\ P_2G_2$ ... ... (1)

Now $$x_1x_2=\frac{y_1y_2}{16a^2}=4a^2 \quad \ldots \quad (2)$$

$$SP_1.\ SP_2=AS.\ SP \quad \ldots \quad (3)$$

$$P_1G_1.\ P_2G_2=\sqrt{(4a^2+y_1^2)(4a^2+y_2^2)}$$

$$=\sqrt{16a^4+4a^2(4a^2t^2-16a^2)+64a^4}$$

$$=\sqrt{16a^4+16a^4t^2}$$
$$=2a\sqrt{4a^2+4a^2t^2}=2a\sqrt{PN^2+NG^2}$$
$$=2a.\ PG\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (4)$$

From (1), (2), (3) and (4), we get

$$PP_1.\ PP_2=2SP.\ PG$$
$$=PQ.\ AN \quad \text{using (i).}$$

This proves the second part.

(iii) $SG_1=SP_1$; $SG_2=SP_2$.

$\therefore \quad M\hat{S}P_1=\pi=2\theta, \quad M\hat{S}P_2=\pi-2\theta_2.$

$\therefore \quad P_1\hat{S}P_2=2(\theta_2-\theta_1)=2\ P_1SP_2$

=the angle subtended by P_1P_2 at the circumcentre of $\Delta\ P_1PP_2$.

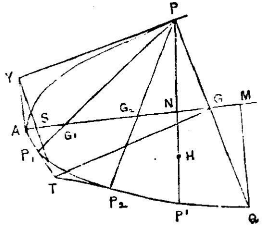
Hence the circle through P_1SP_2 passes through the circumcentre of $\Delta\ PP_1P_2$.

Note by A. A. Krishnaswami Aiyangar, B. A.

Let Y be the foot of the perpendicular from S on the tangent at P; PP′ the double ordinate through P; H the mid-point of NP′, and T the pole of the chord P_1P_2.

1. *The eight points A, Y, P, G, H, P_2, T, P_1 lie on a circle.*

For, the normal at 't' meets the curve again at the point '$-t-\dfrac{2}{t}$' which is evidently unaltered by changing $\dfrac{2}{t}$. Therefore the normals at t, $\dfrac{2}{t}$ meet on the curve at $\left(-t-\dfrac{2}{t}\right)$. Now the tangents at t, $\dfrac{2}{t}$ meet at $2a,\ a\left(t+\dfrac{2}{t}\right)$.



Thus if P_1, P_2, be t and $\frac{2}{t}$, P is $\left(-t-\frac{2}{t}\right)$ and the co-ordinates of T are $2a$, $a\left(t+\frac{2}{t}\right)$, so that if TL is drawn perpendicular to AS, TL= $\frac{1}{2}$ PN=NM or AY. That is the angle THP is a right angle and AL= 2AS, proving that YS passes through T and YS=ST.

∴ PG=2YS=YT and PGTY is a rectangle.

Since angles YAG, PP_1T, TP_2P, and THP are right angles, the circle PGTY passes through A, P_1, P_2, H.

(2) *The centre of this circle lies on the circle* SP_1P_2.

For P_1SP_2=double of the supplement of $P_1\hat{T}P_2$.

=double of $P_1\hat{P}P_2$

$=P_1\hat{X}P_2$, where X is the centre of the circle PP_1P_2.

Hence P_1, S, X, P_2 are concyclic.

Also, by elementary geometry, it is easily seen that SX cuts the circle PP_1P_2 in Y′ and Z′, which are the excentres of the triangle SP_1P_2.

Thus the circle PP_1P_2 passes through *seven* additional points, *viz.*: A, Y, G, H, T, and the excentres of the triangle SP_1P_2, opposite to P_1, P_2.

(3) Because AYGT is cyclic, therefore

$$AS.SP=AS.SG=ST^2=SP_1.SP_2.$$

Question 560.

(N. Sankara Aiyar M.A.) :—Find the volume contained between the two ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ and } \frac{x^2}{b^2}+\frac{z^2}{c^2}+\frac{y^2}{a^2}=1.$$

Solution by T. P. Trivedi, M.A., LL.B., and T. P. Bhaskarasastri, B.A.

Suppose $a>b$; the area common to the two ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \text{ is given by}$$

$$4\left\{\frac{b}{a}\int_0^{\frac{ab}{\sqrt{a^2+b^2}}}\sqrt{a^2-x^2}\,dx+\frac{a}{b}\int_{\frac{ab}{\sqrt{a^2+b^2}}}^{b}\sqrt{b^2-x^2}\,dx\right\}$$

[This is obvious on a little consideration; the ellipses intersect at points $\pm\frac{ab}{\sqrt{a^2+b^2}}, \pm\frac{ab}{\sqrt{a^2+b^2}}$ and the common area may be represented as the sum of two integrals.]

On simplifying, we find the common area to be

$$4\left\{ \tfrac{1}{2}\, ab \sin^{-1}\frac{b}{\sqrt{a^2+b^2}} + \tfrac{1}{2}\, ab \left(\frac{\pi}{2} - \sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\right) \right\}$$

$$= 4\, ab \sin^{-1}\frac{b}{\sqrt{a^2+b^2}}.$$

If we take the sections of the ellipsoids by a plane parallel to the plane of xy and at a distance z from it, we get the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z^2}{c^2} \text{ and } \frac{x^2}{b^2}+\frac{y^2}{a^2}=1-\frac{z^2}{c^2},$$

and the area common to these two is

$$4ab\left(1-\frac{z^2}{c^2}\right)\sin^{-1}\frac{b}{\sqrt{a^2+b^2}}$$

The volume common to tha two ellipsoids

$$=\int_{-c}^{c} 4ab\left(1-\frac{z^2}{c^2}\right)\sin^{-1}\frac{b}{\sqrt{a^2+b^2}}\,dz.$$

$$=\frac{16\, abc}{3}\sin^{-1}\frac{b}{\sqrt{a^2+b^2}}.$$

If $b=a$, the ellipsoids coincide and we get $\frac{4\pi a^2 c}{3}$ for the common volume, as is otherwise obvious.

Question 566.

(S. Krishnaswami Aiyangar) :—Prove that

$$\sin^{-1}x=\frac{x}{(1-x^2)^{\frac{3}{2}}}\left\{1-s_2\frac{x^2}{1-x^2}+s_3\left(\frac{x^2}{1-x^2}\right)^2-\dots\dots\dots\right\}$$

where s_r=the sum of the reciprocals of the first r odd numbers.

Solution by J. C. Swaminarayan, M.A., K. T. Sundara Rajan, B.A., R. J. Pocock B.A., B.Sc., F.R.A.S., and A. Narasinga Rao.

Now $$\frac{\tan^{-1}z}{1+z^2}=\left(z-\frac{z^3}{3}+\frac{z^5}{5}-\dots\dots\dots\right)\left(1-z^2+z^4-\dots\dots\dots\right)$$

$$=z-(1+\tfrac{1}{3})z^3+(1+\tfrac{1}{3}+\tfrac{1}{5})z^5-\dots\dots\dots$$

$$=z-s_2z^3+s_3z^5-\dots\dots\dots$$

$\therefore$ $$\tan^{-1}z=z(1+z^2)\left\{1-s_2z^2+s_3z^4-\!-\dots\dots\right\}$$

$\therefore$ $$\sin^{-1}x=\tan^{-1}\left\{\frac{x^2}{\sqrt{1-x^2}}\right\}$$

$$=\frac{x}{(1-x^2)^{\frac{3}{2}}}\left\{1-s_2\frac{x^2}{1-x^2}+s_3\left(\frac{x}{1-x^2}\right)^2-\dots\dots\dots\right\}$$

Question 568.

(K. V. Anantanarayana Sastri, B. A.) :—Evaluate

$$\int_0^\infty \frac{\log x}{1+x^4}\,dx.$$

Solution (1) by R. J. Pocock, J. C. Swaminarayan, T. P. Trivedi, M. A., LL. B. and (2) by J. M. Bose, M. A., B. Sc.

(1) We have the integral

$$u=\int_0^\infty \frac{x^m dx}{1+x^n} \text{ (where } m<n)=\frac{\pi}{n} \operatorname{cosec} \frac{(m+1)\pi}{n}.$$

[Williamson p. 141.]

Differentiating for m,

$$\int_0^\infty \frac{x^m \log x\, dx}{1+x^n}=-\frac{\pi^2}{n^2}.\operatorname{cosec} \frac{(m+1)\pi}{n} \cot \frac{(m+1)\pi}{n}.$$

If $m=0$, $\displaystyle\int_0^\infty \frac{\log x dx}{1+x^n}=-\frac{\pi^2}{n^2} \operatorname{cosec} \frac{\pi}{n} \cot \frac{\pi}{n}.$ ... ... (1)

If $n=4$, $\displaystyle\int_0^\infty \frac{\log x dx}{1+x^4}=-\frac{\pi^2}{16} \operatorname{cosec} \frac{\pi}{4} \cot \frac{\pi}{4}=-\frac{\pi^2}{8\sqrt{2}}.$

If we differentiate r times for m, we get the integral

$$\int_0^\infty \frac{x^m (\log x)^r dx}{1+x^n}=\left(\frac{\pi}{n}\right)^{r+1} \frac{d^r}{d\theta^r} (\operatorname{cosec} \theta)$$

where $\theta=\dfrac{(m+1)\pi}{n}$.

(2) It will be perhaps more interesting if we work out the more general case

$$\int_0^\infty \frac{\log x}{1+x^n}dx$$

n being a positive integer.

On putting $x^n=v$ and $\dfrac{1}{n}=\alpha$ $(0<\alpha<1)$

$$\int_0^\infty \frac{\log x}{1+x^n}dx=\frac{1}{n^2}\int_0^\infty \frac{v^{\alpha-1}\log v}{1+v}dv.$$

Now assume

$$I=\int_0^\infty \frac{x^{\alpha-1}\log x}{1+x}dx,\ J=\int_0^\infty \frac{x^{\alpha-1}\log x}{1-x}dx.$$

Take a contour consisting of the part of the real axis from $-\infty$ to $+\infty$, two small semicircles γ' and γ of radii r' and r round the

points $x=-1$, and the origin respectively, and a semicircle Γ of large radius, each on the upper half of the z—plane. Let the integral

$$\int_c \frac{z^{\alpha-1}\log z}{1+z}dz$$

be taken round this contour. As the integrand has no singularities within

$$\int_c = \int_0^\infty + \int_\Gamma + \int_{-\infty}^{-1-r'} - \int_{\Upsilon'} + \int_{-1+r'}^{0} - \int_\Upsilon = 0 \quad \ldots \quad (1)$$

At all points on Γ, z is very large and the integrand is sensibly equal to $\frac{\log z}{z^{2-\alpha}}$ which vanishes in the limit.

$$\therefore \qquad \int_\Gamma = 0.$$

At all points on Υ, z is small, and the integrand is sensibly equal to $z^{\alpha-1}\log z$ so that $\int z^{\alpha-1}\log z dz = \left[z^\alpha\left(\frac{1}{\alpha}-\frac{1}{\alpha^2}\right)\right]_\Upsilon$ which vanishes when the radius of Υ is sufficiently diminished.

Hence $\int_\Upsilon = 0$; so that from (1)

$$\int_0^\infty dx \frac{x^{\alpha-1}\log x}{1+x}\left(\int_{-\infty}^{-1-r'} + \int_{-1+r'}^{0}\right)\frac{z^{\alpha-1}\log z\ dz}{1+z} = \Upsilon' \int_{\Upsilon'}.$$

Now $$\int_{\Upsilon'} = \pi i \times \text{residue of } \frac{z^{\alpha-1}\log z}{1+z} \text{ at } z=-1$$

$$= -\pi^2(-1)^{\alpha-1}.$$

Again within the range $-\infty$ to 0, z is real and negative to that on putting $z=-x$.

$$\left(\int_{-\infty}^{-1-r'} + \int_{-1+r'}^{0}\right)\frac{z^{\alpha-1}\log z}{1+z}dz$$

$$= \left(\int_\infty^{1+r'} + \int_{1-r'}^{0}\right)\frac{(-1)^{\alpha-1}x^{\alpha-1}(\log x + \pi i)}{1-x}dx$$

$$= (-1)^{\alpha-1}\left(\int_\infty^{1+r'} + \int_{1-r'}^{0}\right)\frac{x^{\alpha-1}\log x}{1-x}dx$$

$$+ (-1)^{\alpha-1}\pi i\left(\int_\infty^{1+r'} + \int_{1-r'}^{0}\right)\frac{x^{\alpha-1}}{1-x}dx.$$

Now in the limit when r' is sufficiently diminished

$$\int_{\infty}^{1+r'} + \int_{1-r'}^{0} = \int_{0}^{\infty} \text{ and } \int_{0}^{\infty} \frac{x^{\alpha-1}}{1-x}\,dx = \pi \cot \alpha\pi.$$

(Williamson p. 141.)

So that finally

$$\int_{0}^{\infty} \frac{x^{\alpha-1}\log x}{1+x}\,dx + \mathrm{J}(-1)^{\alpha-1}$$

$$+\pi^2 i(-1)^{\alpha-1}\cot \alpha\pi = -\pi^2(-1)^{\alpha-1};$$

or $$(-1)^{1-\alpha}\mathrm{I}+\mathrm{J}+\pi^2 i \cot \alpha\pi = -\pi^2.$$

Equating imaginary parts

$$\mathrm{I}\sin\frac{\pi}{n} = -\pi^2 \cot\frac{\pi}{n}.$$

$$\therefore \quad \int_{0}^{\infty} dx\,\frac{\log x}{1+x^n} = -\frac{\pi^2}{n^2}\,\frac{\cos\frac{\pi}{n}}{\sin^2\frac{\pi}{n}}.$$

Putting $n=4$

$$\int_{0}^{\infty} \frac{\log x}{1+x^4}\,dx = -\frac{\pi^2}{8\sqrt{2}}.$$

Question 570.

(Professor Sanjana) :—Prove that when n and k are positive integers, the sums of the following series vanish :—

(a) $$1-\binom{n+2k}{2}\frac{1}{2n+2k-1}+\binom{n+2k}{4}\frac{1.3}{(2n+2k-1)(2n+2k-3)}$$

$$-\binom{n+2k}{6}\frac{1.3.5}{(2n+2h-1)(2n+2k-3)(2n+2k-5)}+\ldots\ldots\ldots$$

(b) $$\binom{2n+2k+1}{1}-\binom{2n+2k+1}{3}\frac{2n+3}{2n+2k-1}$$

$$+\binom{2n+2k+1}{5}\frac{(2n+3)(2n+5}{(2n+2k-1)(2n+2k-3)}-\ldots\ldots\ldots$$

where $\binom{x}{y}$ denotes $x\mathrm{C}y$.

Solution by J. C. Swaminarayan, M.A.

(a) Now $\frac{d^{n+2k}}{dx^{n+2k}}(x^2)^{k-\frac{1}{2}}$ vanishes for all positive integral values of

n and k.

But $\dfrac{d^{n+2k}}{dx^{n+2k}} = \left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\left(k-\frac{5}{2}\right)\cdots\cdots\left(-n-k+\frac{1}{2}\right)\times$

$$(x^2)^{-n-2k-\frac{1}{2}}(2x)^{n+2k}$$

where S stands for $\left[1+\dfrac{(n+2k)(n+2k-1)}{\underline{|1}}\times\dfrac{1}{2^2}\times\dfrac{1}{(-n-k+\frac{1}{2})}\right.$

$$+\frac{(n+2k)(n+2k-1)(n+2k-2)(n+2k-3)}{\underline{|2}}\times\frac{1}{2^4}$$

$$\times\frac{1}{(-n+k+\frac{1}{2})(-n-k+\frac{3}{2})}.$$

The series inside the brackets can easily be put in the form

$$1-\binom{n+2k}{2}.\ \frac{1}{2n+2k-1}+\binom{n+2k}{4}.\ \frac{1.3}{(2n+2k-1)(2n+2k-3)}$$

$$-\binom{n+2k}{6}\ \frac{1.3.5}{(2n+2k-1)(2n+2k-3)(2n+2k-5)}+\cdots\cdots$$

whence the result (a) follows at once.

(b) Dividing both the sides by $(2n+2k+1)$, we have to prove that

$$1-\frac{2n+2k}{\underline{|3}}(2n+3)+\frac{(2n+2k)(2n+2k-2)}{\underline{|5}}(2n+3)(2n+5)-\ldots\ldots=0,$$

or $$1-(n+k)\frac{2n+3}{3}+\frac{(n+k)(n+k-1)}{\underline{|2}}\times\frac{(2n+3)(2n+5)}{3.5}-\ldots\ldots=0.$$

Now $\dfrac{d^{n+k}}{dx^{n+k}}\left\{x^{-n-\frac{3}{2}}x^{n+k+\frac{1}{2}}\right\}$ or $\dfrac{d^{n+k}}{dx^{n+k}}\left\{x^{k-1}\right\}$ vanishes for all positive integral values of n and k.

The former is

$$x^{-n-\frac{3}{2}}(n+k+\tfrac{1}{2})(n+k-\tfrac{1}{2})(nk+-\tfrac{3}{2})\ldots\ldots\tfrac{3}{8}x^{\frac{1}{2}}$$

$$\times\left[1+{}_{n+k}C_1\frac{(-n-\frac{3}{2})}{(\frac{3}{2})}+{}_{n+k}C_2\frac{(-n-\frac{3}{2})(-n-\frac{5}{2})}{(\frac{3}{2})(\frac{5}{2})}+\ldots\ldots\right]=0.$$

$$\therefore\qquad 1-\binom{n+k}{1}\frac{2n+3}{3}+\binom{n+k}{2}\frac{(2n+3)(2n+5)}{3.5}-\ldots\ldots\Big]=0,$$

whence the given result (b) follows at once.

Question 571.

(S. Ramanujan) :—If $\dfrac{\pi\alpha}{2}=\log\tan\left(\dfrac{\pi}{4}+\dfrac{\pi\beta}{4}\right)$, shew that

$$\left(\frac{1^2+\alpha^2}{1-\beta^2}\right)\left(\frac{3^2-\beta^2}{3^2+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)\ldots\ldots\ldots=e^{\frac{1}{2}\pi\alpha\beta}.$$

Solution by J. C. Swaminarayan, M.A., R. J. Pocock, B.A., B.Sc., F.R.A.S., A. Narasinga Rao and K. B. Madhava.

By expressing $1-\sin\frac{1}{2}\pi x$ in factors and taking the logarithmic differential co-efficient, we obtain

$$\frac{\pi}{2}\left\{\sec\frac{\pi}{2}x+\tan\frac{\pi}{2}x\right\}=\frac{2}{1-x}-\frac{2}{3+x}+\frac{2}{5-x}-\frac{2}{7+x}+\ldots\ldots$$

and $$\frac{\pi}{2}\left\{\sec\frac{\pi}{2}x-\tan\frac{\pi}{2}x\right\}=\frac{2}{1+x}-\frac{2}{3-x}+\frac{2}{5+x}-\frac{2}{7-x}+\ldots\ldots$$

$$\therefore\quad \frac{\pi x}{2}\sec\frac{\pi}{2}x=\frac{2x}{1-x^2}-\frac{3\cdot 2x}{3^2-x^2}+\frac{5\cdot 2x}{5^2-x^2}-\frac{7\cdot 2x}{7^2-x^2}+\ldots\ldots$$

$$\therefore\quad \int\frac{\pi x}{2}\sec\frac{\pi}{2}x\,dx=-\log(1-x^2)+3\log(3^2-x^2)-5\log(5^2-x^2)+\ldots$$

Changing x into iy, we get

$$\int\frac{\pi y}{2}\operatorname{sech}\frac{\pi y}{2}dy=\log(1+y^2)-3\log(3^2+y^2)+5\log(5^2+y^2)-\ldots$$

The given condition is equivalent to $\cosh\frac{\pi\alpha}{2}=\sec\frac{\pi\beta}{2}$.

$$\therefore\quad \tanh\frac{\pi\alpha}{2}\,d\alpha=\tan\frac{\pi\beta}{2}\,d\beta.$$

$$\therefore\quad d\alpha=\sec\frac{\pi\beta}{2}\,d\beta \text{ and } d\beta=\operatorname{sech}\frac{\pi\alpha}{2}\,d\alpha.$$

$$\therefore\quad \frac{\pi}{2}\alpha\beta=\int\frac{\pi}{2}(\beta d\alpha+\alpha d\beta)$$

$$=\int\frac{\pi}{2}\alpha\operatorname{sech}\frac{\pi\alpha}{2}\,d\alpha+\int\frac{\pi}{2}\beta\sec.\frac{\pi}{2}d\beta$$

$$=\log\frac{1+\alpha^2}{1-\beta^2}-3\log\frac{3^2+\alpha^2}{3^2-\beta^2}+5\log\frac{5^2+\alpha^2}{5^2-\beta^2}-\ldots$$

Hence $$\left(\frac{1+\alpha^2}{1-\beta^2}\right)\left(\frac{3^2-\beta^2}{3^2+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)^5\ldots\ldots\ldots=e^{\frac{1}{2}\pi\alpha\beta}.$$

Question 572.

(M. Bhimansena Rao) :—If $f(x)=x+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\frac{x^7}{7^2}\ldots\ldots\ldots\ldots$, shew that

(i) $f(\sqrt{2}-1)=\frac{\pi^2}{16}-\frac{1}{4}\left\{\log(\sqrt{2}-1)\right\}^2$,

(ii) $f\left(\frac{\sqrt{5}-1}{2}\right)=\frac{\pi^2}{12}-\frac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$,

(iii) $f(\sqrt{5}-2)=\frac{\pi^2}{24}-\frac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$.

Solution by J. C. Swaminarayan, M.A.

It is obvious that $f(0)=0$ and $f(1)=\frac{\pi^2}{8}$.

$$(1)\ f(\sqrt{2}-1)=\int_0^{\sqrt{2}-1}\frac{\tanh^{-1}x}{x}dx$$

$$=\left\{\tanh^{-1}x\log x\right\}_0^{\sqrt{2}-1}-\int_1^{\sqrt{2}-1}\frac{\log x}{1-x^2}dx-\int_0^1\frac{\log x}{1-x^2}dx$$

$$=\frac{\pi^2}{8}-\frac{1}{2}\left\{\log(\sqrt{2}-1)\right\}^2-\int_1^{\sqrt{2}-1}\frac{\log x}{1-x^2}dx.$$

If we put $x=\frac{1-z}{1+z}$, $\int_1^{\sqrt{2}-1}\frac{\log x}{1-x^2}dx$ becomes $\int_0^{\sqrt{2}-1}\frac{\tanh^{-1}z}{z}dz$ or $f(\sqrt{2}-1)$.

$$\therefore\ 2f(\sqrt{2}-1)=\frac{\pi^2}{8}-\frac{1}{2}\left\{\log(\sqrt{2}-1)\right\}^2,$$ whence the result.

$$\text{(ii)}\ \int_0^{\frac{\sqrt{5}-1}{2}}\frac{\log(1+x)}{x}=\left\{\log(1+x)\log x\right\}_0^{\frac{\sqrt{5}-1}{2}}-\int_0^{\frac{\sqrt{5}-1}{2}}\frac{\log x}{1-x}dx$$

$$=-\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2+\int_0^{\frac{\sqrt{5}-1}{2}}\frac{\log(1-z)}{z}dz-\int_0^{\frac{\sqrt{5}-1}{2}}dz,\ \text{where } x+1=\frac{1}{z}$$

$$=-\frac{3}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}+\int_0^{\frac{\sqrt{5}-1}{2}}\frac{\log(1-x)}{x}dx+\frac{\pi^2}{6}.$$

$$\therefore\ \int_0^{\frac{\sqrt{5}-1}{2}}\frac{1}{x}\log\frac{1+x}{1+x}dx=2\int_0^{\frac{\sqrt{5}-1}{2}}\frac{\tanh^{-1}x}{x}dx$$

$$=\frac{\pi^2}{6}-\frac{3}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$

whence the result.

(iii) $$f(\sqrt{5}-2)=\int_0^{\sqrt{5}-2}\frac{\tanh^{-1}x}{x}\,dx=\left\{\tanh^{-1}x\log x\right\}_0^{\sqrt{5}-2}$$

$$-\int_1^{\sqrt{5}-2}\frac{\log x}{1-x^2}\,dx-\int_0^1\frac{\log x}{1-x^2}\,dx.$$

$$=\frac{\pi^2}{8}-\frac{3}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2-\int_1^{\sqrt{5}-2}\frac{\log x}{1-x^2}\,dx.$$

By putting $x=\frac{1-z}{1+z}$, $\int_1^{\sqrt{5}-2}\frac{\log x}{1-x^2}\,dx$ becomes $\int_0^{\frac{\sqrt{5}-1}{2}}\frac{\tanh^{-1}z}{z}\,dz$

or $f\left(\frac{\sqrt{5}-1}{2}\right)$.

$$\therefore f\left(\sqrt{5}-2\right)=\frac{\pi^2}{8}-\frac{3}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2-\frac{\pi^2}{12}+\frac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$

$$=\frac{\pi^2}{24}-\frac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2.$$

Question 573.

(M. Bhimasena Rao) :—Shew that

$$\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^5}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^7}+\ldots\ldots=\frac{\pi^2}{12}-\frac{3}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$

Solution by J. C. Swaminarayan, M.A.

Since

$$\frac{\tanh^{-1}x}{\sqrt{1+x^2}}=x-\frac{2}{3}x^3+\frac{2\cdot4}{3\cdot5}x^5-\frac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7+\ldots\ldots$$

the given series is equal to

$$\int_0^{\frac{1}{2}}\frac{\tanh^{-1}x}{x\sqrt{1+x^2}}\,dx=\int_0^{\frac{1}{2}}\frac{2\tanh^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)}{x\sqrt{1+x^2}}\,dx$$

$$=\int_0^{\sqrt{5}-2}\frac{2\tanh^{-1}z}{z}\,dz,\text{ where }\frac{\sqrt{1+x^2}-1}{x}=z.$$

$$=2\left\{\frac{\pi^2}{24}-\frac{3}{4}\left(\log\frac{\sqrt{5}-1}{2}\right\}^2\right\}\text{ by (iii) of Q. 572.}$$

$$=\frac{\pi^2}{12}-\frac{3}{2}\left(\log\frac{\sqrt{5}-1}{2}\right)^2.$$

Question 578.

(N. P. Pandya) :—Construct a triangle ABC having given the side BC, the distance between the circumcentre and the symmedian point, and also the distance between the incentre and the excentre.

Solution by V. V. S. Narayan.

The circumcircle bisects II_1, which is a diameter of the cyclic quadrilateral $BICI_1$.

Therefore take BC and describe a $\triangle BDC$ so that $BD = CD = \frac{1}{2} II_1$.

Then, the circle BDC will be the circumcircle and its centre O the circumcentre. Hence R is known and since OK is given, w, the Brocard angle of the triangle can be found geometrically from the well known relation

$$OK^2 = R^2(1 - 3 \tan^2 w).$$

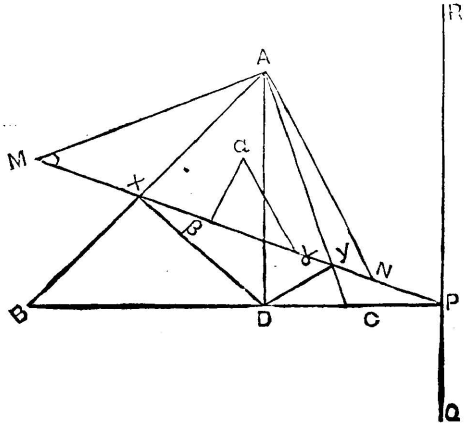
With the usual notation, we know that :—

$\Omega O K = w$, the Brocardangle

and $O \Omega K =$ a right angle.

$\therefore$ $O\Omega = OK \cos w$.

From this relation $O\Omega$ can be easily found geometrically, since OK is given and since w can be determined as given above. Then, draw $B\Omega$ so that $\Omega \hat{B} C = w$. With centre O and radius $O\Omega$ draw an arc to cut $B\Omega$ at Ω. Join $C\Omega$, draw CA making $\Omega \hat{C} A$; equal to w and cutting the circumcircle at A. Then ABC will be the required triangle.



QUESTIONS FOR SOLUTION

613. (P. V. SESHU AIYAR):—Show that

$$\operatorname*{Lt}_{z=0} \frac{\dfrac{\Pi(s)}{\Pi(s+z)}\left(\dfrac{x}{2}\right)^{z} - \dfrac{\Pi(n+s)}{\Pi(n+s-z)}\left(\dfrac{x}{2}\right)^{-z}}{z}$$

$$=2\log\frac{x}{2}-\psi(s)-\psi(n+s),$$

where $\psi(x)$ stands for $\dfrac{d}{dx}\log \Pi(x)$.

614. (M. T. NARANIENGAR):—If ${}_1u_n=\left(\dfrac{1.3.5.......2n-1}{2.4.6.......2n}\right)^2$, prove that

$$u_n=4^{2n}u_{2n}-(4n-2)4^{2n-1}u_{2n-1}+\frac{(4n-3)(4n-4)}{1.2}4^{2n-2}u_{2n-2}-\ldots\ldots$$

615. (SELECTED):—Find two numbers such that the sum of all the aliquot parts of either is equal to the other.

616. (S. NARAYANA AIYAR, M.A.):—Establish the following extension of Maclaurin's theorem:—

$$\phi(x)+\frac{a-b}{b}\cdot\frac{x}{1}\phi'(x)+\frac{(a-b)(a-b-1)}{b.(b+1)}\cdot\frac{x^2}{1\cdot2}\phi''(x)+$$

$$\frac{(a-b)(a-b-1)(a-b-2)}{b.(b+1)(b+2)}\cdot\frac{x^3}{1\cdot2\cdot3}\phi'''(x)+\ldots$$

$$=\phi(0)+\frac{a}{b}x\phi'(0)+\frac{a(a+1)}{b(b+1)}\cdot\frac{x^2}{1\cdot2}\phi''(0)+$$

$$\frac{a(a+1)(a+2)}{b(b+1)(b+2)}\cdot\frac{x^3}{1\cdot2\cdot3}\phi'''(0)+\ldots\ldots$$

(a) When $a=b$, this reduces to Maclaurin's theorem.
(b) When $\phi(x)=e^x$, this reduces to Question 576 (I. M. J.)
(c) Examine the cases when $\phi(x)=\cos x$, $(1+x)^{-n}$, etc.

617. (S. NARAYANA AIYAR, M.A.):—If $F(\alpha, \beta, \gamma, x)$ denotes the hyper-geometric series $1+\dfrac{\alpha.\beta}{\gamma.1}x+\dfrac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot1\cdot2}x^2+\ldots\ldots$, shew that

$$\sum_{r=\infty}^{r=0}\left[\frac{\Gamma(b)\Gamma(\gamma)\Gamma(a-b+1)\Gamma(\alpha+r)\Gamma(\beta+r)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(b+r)\Gamma(\gamma+r)\Gamma(a-b-r+1)}\cdot\frac{x^r}{\Gamma(r+1)}\right.$$

$$\left.F(\alpha',\beta',\gamma',x)\right]$$

$$=1+\frac{a}{b}\cdot\frac{\alpha\cdot\beta}{\gamma\cdot 1}x+\frac{a(a+1)\alpha(\alpha+1)\cdot\beta(\beta+1)}{b(b+1)\gamma(\gamma+1)1.2}x^2+\ldots$$

where $\alpha'=\alpha+r$, $\beta'=\beta+r$, $\gamma'=\gamma+r$.

Examine the case when $\alpha=\gamma$.

618. (Professors K. J. SANJANA and T. P. TRIVEDI):—Establish the following identities, n being a positive integer and p any rational number $>2n$:

(1) $$1-\frac{2n-1}{2^2\lfloor 1}\,\frac{p+2n+1}{2n-1}+\frac{(2n-2)(2n-3)}{2^4\lfloor 2}\,\frac{(p+2n+1)(p+2n-1)}{(2n-1)(2n-3)}$$
$$-\frac{(2n-3)(2n-4)(2n-5)}{2^6\lfloor 3}\,\frac{(p+2n+1)(p+2n-1)(p+2n-3)}{(2n-1)(2n-3)(2n-5)}+\ldots$$
$$=\frac{(-1)^n}{2^{2n}}\,\frac{(p-1)(p-3)\ldots\ldots(p-2n+1)}{1.3.5\ldots(2n-3)(2n-1)};$$

(2) $$1-\frac{2n}{2^2\lfloor 1}\,\frac{p+2n}{2n-1}+\frac{2n(2n-3)}{2^4\lfloor 2}\,\frac{(p+2n)(p+2n-2)}{(2n-1)(2n-3)}$$
$$-\frac{2n(2n-4)(2n-5)}{2^6\lfloor 3}\,\frac{(p+2n)(p+2n-2)(p+2n-4)}{(2n-1)(2n-3)(2n-5)}+\ldots$$
$$=\frac{(-1)^n}{2^{2n-1}}\,\frac{p(p-2)\ldots(p-2n+)2}{1.3.5-(2n-3)(2n-1)};$$

(3) $$1-\frac{(2n+1)^2-1^2}{\lfloor 3}\,\frac{p+2}{p+3}\,\frac{\{(2n+1)^2-1^2\}}{}$$
$$+\frac{\{(2n+1)^2-3^2\}}{\lfloor 5}\,\frac{(p+2)(p+4)}{(p+3)(p+5)}-\ldots$$
$$=\frac{(-1)^n}{2n+1}\,\frac{(p-1)(p-3)+\ldots(p-2n+1)}{(p+3)(p-5)-\ldots(p+2n+1)};$$

(4) $$1-\frac{(2n+1)^2-1^2}{\lfloor 2}\,\frac{1}{p+3}$$
$$+\frac{\{(2n+1)^2-1^2\}\{2n+1)^2-3^2\}}{\lfloor 4}\,\frac{1\cdot 3}{(p+3)(p+5)}-\ldots\ldots$$
$$=\frac{(p-1)(p-3)\ldots(p-2n+1)}{(p+3)(p+5)\ldots(p+2n+1)},$$

(5) $$1-\frac{(2n)^2}{\lfloor 2}\,\frac{p+1}{p+2}+\frac{(2n)^2\{(2n)^2-2^2\}}{\lfloor 4}\,\frac{(p+1)(p+3)}{(p+2)(p+4)}-\ldots\ldots$$
$$=-(1)^n\frac{p(p-2)\ldots(p-2n+2)}{(p+2)(p+4)\ldots(p+2n)},$$

(6) $1-\frac{(2n)^2}{\lfloor 2}\frac{1}{p+2}+\frac{(2n)^2\{(2n)^2-2^2\}}{\lfloor 4}\frac{1\cdot 3}{(p+2)(p+4)}-\dots$

$$=\frac{p(p-2)\dots(p-2n+2)}{(p+2)(p+4)\dots(p+2n)}.$$

[Each of the series *terminates at the* $(n+1)$ *th* term.]

619. (S. Krishnaswami Aiyangar):—

If $s_n=\frac{1}{(2n-1)^2(2n-3)^2}+\frac{1}{(2n-5)^2}\left\{\frac{1}{2n-1)^2}+\frac{1}{(2n-3)^2}\right\}$

$$+\frac{1}{(2n-7)^2}\left\{\frac{1}{(2n-1)^2}+\frac{1}{(2n-3)^2}+\frac{1}{(2n-5)^2}\right\}+\dots$$

shew that

$$\frac{s_2}{5^2}+\frac{s_3}{7^2}+\frac{s_4}{9^2}+\quad=\frac{\pi^6}{2^6\lfloor 6}.$$

620. (S. Krishnaswami Aiyangar):—Find the value of

$$1-\frac{3^5x^2}{2\,!}+\frac{5^5x^4}{4\,!}-\frac{7^5x^6}{6\,!}+\dots\dots\dots$$

621. (R. Vythynathaswamy):—If C is the chord of an arc λ of a closed curve, shew that $\int Cds$ taken over the contour is equal to λ times the perimeter of the locus of centroids of arcs of the curve equal to λ.

622. (R. Vythynathaswamy):—A moving quadrilateral has its four corners on a fixed circle. Shew that at any instant the sum of the angular velocities of the four pedal lines is equal to the sum of the angular velocities of the corners about the centre.

623. (M. Bhimasena Rao):—Two triangles in perspective are such that the perpendiculars from the vertices of either, on the corresponding sides of the other, are concurrent. Shew that the two points of concurrence of these perpendiculars are in a line with the centre of perspective.

624. (A. N. Raghavachar):—Two parabolas of latera recta $4a$, $4b$ have a common vertex and their axes are inclined at an angle α. If they again intersect at an angle $(\alpha-\beta)$ and if the distance between their foci is c, prove that

$$\frac{c^2}{ab\cos\alpha}=\frac{9t^2s(3ts^2-4t+8s)}{4(t-2s)^2(2t-s)}.$$

625. (V. V. S. NARAYAN):—Shew that the lines joining the vertices of a triangle to the corresponding vertices of its first Brocard triangle concur in a point Q whose trilinear co-ordinates are inversely proportional to those of the pole of $\Omega\,\Omega'$ with respect to the Brocard circle.

626.—(M. BHIMASENA RAO):—If $f(x)$ denotes

$$\operatorname{sech}\frac{\pi x}{2}-\frac{1}{3}\operatorname{sech}\frac{3\pi x}{2}+\frac{1}{5}\operatorname{sech}\frac{5\pi x}{2}-\frac{1}{7}\operatorname{sech}\frac{7\pi x}{2}+\ldots$$

shew that

(i) $f(1)=\frac{\pi}{8}$

(ii) $f(\sqrt{3})=\frac{\pi}{24}$

(iii) $f(\sqrt{7})=\frac{1}{2}\left(\sin^{-1}\frac{3}{4}-\frac{\pi}{4}\right)$

(iv) $f(\sqrt{15})=\frac{1}{2}\left(\frac{\pi}{12}-\sin^{-1}\frac{1}{4}\right)$

(v) $f\left(\sqrt{\frac{3}{5}}\right)=\frac{1}{2}\left(\cos^{-1}\frac{1}{4}-\frac{\pi}{12}\right)$

(vi) $f(x)+f(x^{-1})=\frac{\pi}{4}$.

Are there any values of x other than zero and those given by (vi) for which $f(x)$ can be evaluated in finite terms?

627. (K. V. ANANTANARAYANA SASTRI, B. A.):—Four spheres of radii a, b, c, d, intersect at right angles. Shew that the volume of the tetrahedron formed by their centres is

$$\frac{1}{6}abcd(a^{-2}+b^{-2}+c^{-2}+d^{-2})^{\frac{1}{2}}.$$

628. (K. V. ANANTANARAYANA SASTRI, B. A.):—Prove that the length of the first negative pedal of the loop of the Folium of Descartes $x^3+y^3-3axy=0$ is $6a-a(\pi-\sqrt{2}\log 1+\sqrt{2})$.

629. (S. Ramanujan) :—Prove that

$$\frac{\frac{1}{2}+\sum_{n=1}^{n=\infty} e^{-\pi n^2 x}\cos(\pi n^2\sqrt{1-x^2})}{\sum_{n=1}^{n=\infty} e^{-\pi n^2 x}\sin(\pi n^2\sqrt{1-x^2})}=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{(1-x)}};$$

and deduce the following:

(a)
$$\frac{\frac{1}{2}+\sum_{n=1}^{n=\infty} e^{-\pi n^2}}{\frac{1}{2}+\sum_{n=1}^{n=\infty} e^{-5\pi n^2}}=\sqrt{5\sqrt{5}-10}.$$

(b)
$$\sum_{n=1}^{n=\infty} e^{-\pi n^2}(\pi n^2-\tfrac{1}{4})=\tfrac{1}{8}.$$